ASYMPTOTIC FORMS OF PHASE FUNCTIONS FOR NONABSORBING, RADIALLY INHOMOGENEOUS PARTICLES

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There has recently been great interest in calculating the light-scattering characteristics of small spherical particles having a radially inhomogeneous index of refraction [1-3]. This problem can be solved analytically only in certain special cases [1, 2], so it becomes necessary to develop numerical methods of solving it [1, 3]. One of them is the method of phase functions [4], enabling one to calculate fairly accurately the optical characteristics of radially inhomogeneous particles using relatively little computer time and opening up broad prospects for investigating the optics of such particles. Some problems associated with this method are still unsolved, however. No expressions have been found, in particular, to describe the behavior of phase functions near the center of a sphere, the use of which would enhance the accuracy of the results obtained. Nor are there criteria for estimating the accuracy of calculations by this method. Those gaps are filled in the present paper.

Let us consider nonmagnetic and nonabsorbing spherical particles having a radially dependent index of refraction n(r). We assume that n(r) is a continuously differentiable function. We designate the particle's radius as a, the wave number of the incident radiation as k, and the particle's diffraction parameter as x = ka. According to the method of phase functions, the coefficients of the scattering series [1, 2, 5] are

$$a_l = \frac{\psi_l(x)\,w_l'(x) - n^2(x)\,\psi_l'(x)\,w_l(x)}{\xi_l(x)\,w_l'(x) - n^2(x)\,\xi_l'(x)\,w_l(x)}, \quad b_l = \frac{\psi_l(x)\,g_l'(x) - \psi_l'(x)\,g_l(x)}{\xi_l(x)\,g_l'(x) - \xi_l'(x)\,g_l(x)}, \tag{1}$$

where

$$g_l(x) = \cos\delta_l^c(x)\,\psi_l(x) - \sin\delta_l^c(x)\,\chi_l(x);$$
$$w_l(x) = \cos\delta_l^w(x)\,\psi_l(x) - \sin\delta_l^w(x)\,\chi_l(x);$$
$$g_l'(x) = \cos\delta_l^c(x)\,\psi_l'(x) - \sin\delta_l^c(x)\,\chi_l'(x);$$
$$w_l'(x) = \cos\delta_l^w(x)\,\psi_l'(x) - \sin\delta_l^w(x)\,\chi_l'(x);$$

δ_ℓ^W and δ_ℓ^C are phase functions satisfying the equations

$$\begin{aligned} \frac{d}{d\rho}\delta_l^w &= (n^2(\rho)-1)\left[\cos\delta_l^w(\rho)\,\psi_l(\rho) - \sin\delta_l^w(\rho)\,\chi_l(\rho)\right]^2 - \\ &- [\ln(n^2(\rho))]'\left[\cos\delta_l^w(\rho)\,\psi_l(\rho) - \sin\delta_l^w(\rho)\,\chi_l(\rho)\right]\left[\cos\delta_l^w(\rho)\,\psi_l'(\rho) - \right. \\ &\left. - \sin\delta_l^w(\rho)\,\chi_l'(\rho)\right], \\ \frac{d}{d\rho}\delta_l^c &= (n^2(\rho)-1)\left[\cos\delta_l^c(\rho)\,\psi_l(\rho) - \sin\delta_l^c(\rho)\,\chi_l(\rho)\right]^2 \end{aligned} \tag{2}$$

with the boundary conditions

$$\delta_l^w(0) = \delta_l^c(0) = 0; \tag{3}$$

ψ_ℓ and χ_ℓ are Riccati-Bessel functions; ξ_ℓ are Riccati-Hankel functions of the first kind; $\psi_\ell(\rho) = (\pi\rho/2)^{1/2}J_{\ell+1/2}(\rho)$; $\chi_\ell(\rho) = (\pi\rho/2)^{1/2}N_{\ell+1/2}(\rho)$; $\xi_\ell(\rho) = \psi_\ell(\rho) + i\chi_\ell(\rho)$; $\rho = kr$; r is the current coordinate.

Let us find expressions describing the behavior of phase functions for $\rho \ll \ell$. We write Eqs. (2) in the integral form

$$\delta_l^w(\rho) = \int_0^\rho dz\,(n^2(z)-1)\left[\cos\delta_l^w(z)\,\psi_l(z) - \sin\delta_l^w(z)\,\chi_l(z)\right]^2 -$$

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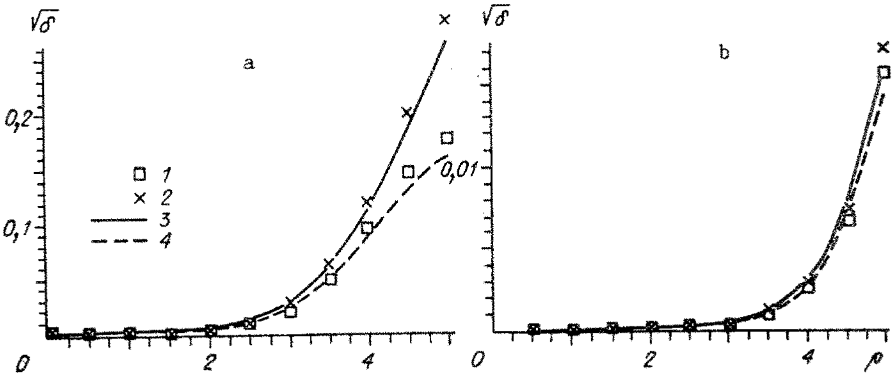

Fig. 1

$$- [\ln (n^2 (z))]' \left[\cos \delta_l^w (z) \psi_l (z) - \sin \delta_l^w (z) \chi_l (z)\right] \left[\cos \delta_l^w (z) \psi_l' (z) - \sin \delta_l^w (z) \chi_l' (z)\right],$$
$$\delta_l^c (\rho) = \int_0^\rho dz \,(n^2 (z) - 1) \left[\cos \delta_l^c (z) \psi_l (z) - \sin \delta_l^c (z) \chi_l (z)\right]^2. \quad (4)$$

For $\rho \ll \ell$, the phases are so small [6] that on the right sides of Eqs. (4) we can retain only the terms containing $\cos \delta_\ell^W$ and $\cos \delta_\ell^c$; they then become

$$\delta_l^w (\rho) = \int_0^\rho dz \,(n^2 (z) - 1) \psi_l^2 (z) - [\ln (n^2 (z))]' \psi_l (z) \psi_l' (z),$$
$$\delta_l^c (\rho) = \int_0^\rho dz \,(n^2 (z) - 1) \psi_l^2 (z). \quad (5)$$

These are the expressions that describe the behavior of phase functions for $\rho \ll \ell$. Let us find the conditions of their applicability. The relations

$$\left|\cos \delta_l^w (z) \psi_l (z)\right| \gg \left|\sin \delta_l^w (z) \chi_l (z)\right|, \quad \left|\cos \delta_l^w (z) \psi_l' (z)\right| \gg \left|\sin \delta_l^w (z) \chi_l' (z)\right|,$$
$$\left|\cos \delta_l^c (z) \psi_l (z)\right| \gg \left|\sin \delta_l^c (z) \chi_l (z)\right|, \quad \left|\cos \delta_l^c (z) \psi_l' (z)\right| \gg \left|\sin \delta_l^c (z) \chi_l' (z)\right| \quad (6)$$

for $0 < \rho < a$ are used in the transition from (4) to (5). We use asymptotic expressions for Riccati–Bessel functions for $\rho \ll \ell$ {[5], Eqs. (5.1) and (5.2)}:

$$\psi_l (\rho) \simeq \frac{\rho}{2\sqrt{2}\, l} \left(\frac{e\rho}{2l}\right)^l, \quad \chi_l (\rho) \simeq -\sqrt{2} \left(\frac{e\rho}{2l}\right)^l. \quad (7)$$

In accordance with (7), the first and second relations in (6) are equivalent to each other, as are the third and fourth. Since the phases are small, the conditions (6) can be written

$$\psi_l (z) \gg \left|\delta_l^w (z) \chi_l (z)\right|, \quad \psi_l (z) \gg \left|\delta_l^c (z) \chi_l (z)\right|. \quad (8)$$

In accordance with (7), for small ρ Eqs. (5) have the form

$$\delta_l^w = (n_0^2 - 1) \frac{\rho^3}{16 (l + 1) l^2} \left(\frac{e\rho}{2l}\right)^{2l} - \frac{n_0'}{n_0} \frac{\rho^2}{8 l^2} \left(\frac{e\rho}{2l}\right)^l,$$
$$\delta_l^c = (n_0^2 - 1) \frac{\rho^3}{16 (l + 1) l^2} \left(\frac{e\rho}{2l}\right)^{2l}, \quad (9)$$

where n_0 and n_0' are the mean values of the index of refraction and its derivative. For $\rho \ll \ell$ we can neglect the second term in the expression for δ_ℓ^W. Using (7) and (9), we write the conditions (8) as

$$\frac{\rho}{2\sqrt{2}\, l} \left(\frac{e\rho}{2l}\right)^l \gg (n_0^2 - 1) \frac{\rho^3 \sqrt{2}}{16^2 l} \left(\frac{e\rho}{2l}\right)^l \quad (10)$$

or

$$(n_0^2 - 1) \frac{\rho^2}{4 l^2} \ll 1. \quad (11)$$

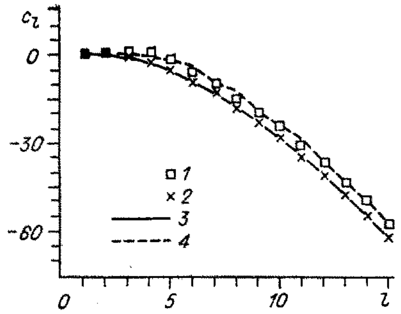

Fig. 2

And this is the condition of applicability of the asymptotic form (5). From (11) it follows, in particular, that near the center of a particle the functions $\delta_\ell^W(\rho)$ and $\delta_\ell^C(\rho)$ can be described by Eqs. (5) for any ℓ.

We calculated $\delta_\ell^W(\rho)$ and $\delta_\ell^C(\rho)$ for radial profiles n(r):

$$n(\rho) = (2 - (\rho/x)^2)^{1/2},\ x = 5{,}0 \text{ (a Luneburg lens [2]);} \tag{12a}$$

$$n(\rho) = 1{,}5/(1 + 0{,}0051\rho^2),\ x = 5{,}0\ [1]. \tag{12b}$$

In Fig. 1a we show the functions $\delta_5^W(\rho)$ and $\delta_5^C(\rho)$ obtained by solving Eqs. (2) (points 1 and 2) and those in accordance with the asymptotic forms (5) (curves 3 and 4) for the profile (12a), and in Fig. 1b we show the functions $\delta_8^W(\rho)$ and $\delta_8^C(\rho)$ for the profile (12b). Comparing the solutions of Eqs. (2) with the asymptotic forms (5), we note fairly good agreement between (5) and $\delta_\ell^W(\rho)$ and $\delta_\ell^C(\rho)$ even for $\rho \simeq 5$, where the condition (11) is violated. For small ρ the asymptotic forms (5) fully coincide with the solutions of Eqs. (2).

Using the asymptotic forms (5) enables one to solve Eqs. (2) more accurately. The point is that numerical integration of Eqs. (2) must begin at $\rho > 0$, since the function $\chi_\ell(\rho)$ is not defined at zero. Choosing the initial values of the phase functions in accordance with (5) enables one to increase the calculation accuracy.

Using the asymptotic forms (5) also makes it possible to monitor the accuracy in calculating the scattered-field coefficients (1) calculated by the method of phase functions or by some other method [1]. For this one must substitute the phases $\delta_\ell^W(x)$ and $\delta_\ell^C(x)$ from (5) into Eqs. (1). If the condition (11) is satisfied in the entire particle, the quantities thus found should agree with the calculated coefficients a_ℓ and b_ℓ. The calculation errors can be estimated from the accuracy with which the coefficients a_ℓ and b_ℓ obtained in the calculations agree with their values in accordance with the asymptotic forms (5).

In Fig. 2 we give the quantity $c_\ell = |a_\ell|^2 + |b_\ell|^2$, calculated for the profiles (12a) and (12b) by the method of phase functions (points 1 and 2) and from Eqs. (5) (curves 3 and 4), as a function of ℓ. For ℓ exceeding the particle diffraction parameter x = 5.0, the curves almost coincide, which confirms the good accuracy of the calculations by the method of phase functions.

We have thus considered the behavior of phase functions that are solutions of Eqs. (2), and have obtained asymptotic expressions (5) describing the behavior of the functions $\delta_\ell^W(\rho)$ and $\delta_\ell^C(\rho)$ for small ρ. The conditions of applicability (11) have been established for the derived asymptotic forms. We show that using Eqs. (5) enables one to increase the accuracy of calculations by the method of phase functions. We also suggested a method of monitoring the accuracy in calculating scattered-field coefficients for particles with a radially inhomogeneous index of refraction.

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CONVERGING SHOCK WAVES IN MEDIA WITH DECREASING DENSITY

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A number of papers have studied the cumulation of shock waves (see, e.g., [1-8]. Here our specific interest is in investigating shock waves propagating into a decreasing density. For concentric convergent shock waves the problem was solved in [7] by the method of Whitham [6], with which one can evaluate the gas parameters on the wave front in the presence of a piston coming from infinity and generating a continuous inflow of energy in the focus region.

In this paper we compare the solution of [7] with the results of an approximate study of instantaneous energy release (by a strong explosion) at the edge of a closed region with decreasing density of the medium toward the center. We also derived similar solutions for propagation of a spherical shock wave convergent toward the decreasing density in two limiting cases: the adiabatic and the isothermal approximations. The latter regime of the process is linked with the stage of motion when radiative energy transfer appreciably affects the distribution of the flow parameters of the medium. In contrast with [7] the similarity study gave us both the law of motion of the wave front, and the distribution of flow parameters behind the front.

1. We consider propagation of a shock wave toward a decreasing geometric section A and decreasing density of the medium ρ_0 for two limiting laws of energy release at its boundary: 1) exit of a steady strong shock wave generated by a piston moving in from infinity; 2) a strong explosion on a perfectly rigid wall bounding a region with variable A and ρ_0. Physically this means that in the first case the time for the shock wave to focus t_* is much less than the time t^0 for the piston to reach the boundary of the region ($t_* \ll t^0$), and in the second case we have $t_* \gg \tau$ (τ is the duration of the energy release).

In Case 1, applying the rule of characteristics from Whitham [6], we can obtain an equation for the speed of the front of a strong shock wave in a region with decreasing A and ρ_0:

$$d\ln\left(D_1\rho_0^{\beta}A^{\eta}\right)/dx = 0. \tag{1.1}$$

Here x is the coordinate of the front, reckoned from the boundary of the region examined; $\eta = 1/[1 + 2/k + \sqrt{2k/(k-1)}]$; $\beta = 1/[2 + \sqrt{2k/(k-1)}]$; and k is the index of a polytropic medium. Then from the wave front speed D_1, the pressure at the front $p_1 \sim \rho_0 D_1^2$, and the shock wave power $W_1 \sim p_1 D_1$, from Eq. (1.1) we obtain the expressions

$$D_1 \sim \rho_0^{-\beta}(x)A^{-\eta}(x),\ p_1 \sim \rho_0^{1-2\beta}(x)\,A^{-2\eta}(x),\ W_1 \sim \rho_0^{1-3\beta}(x)\,A^{-3\eta}(x). \tag{1.2}$$

As k varies in the range 11/9-3 the corresponding values of the exponents are $\beta = 0.188$-0.268, $\eta = 0.148$-0.284. This means that the shock speed D_1 increases continuously as the shock propagates, for any laws of decrese of ρ_0 and A. The pressure p_1 and the power W_1 can either decrease, remain steady, or increase, depending on the combination of describing laws for ρ_0 and A, since in the range of k indicated, $1 - 2\beta > 0$ and $1 - 3\beta > 0$.

In case 2 typical parameters of the problem are: surface density of explosive energy is E_0, test region radius is R_0, initial medium density is $\rho_0(R)$, where $R = x_0 - x$ is the radius of the shock wave front, and x_0 is the coordinate of the focus point. In this formu-

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